

# Dual Concepts of Almost Distance-Regularity and the Spectral Excess Theorem\*

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## Abstract

Generally speaking, ‘almost distance-regular’ graphs share some, but not necessarily all, of the regularity properties that characterize distance-regular graphs. In this paper we propose two new dual concepts of almost distance-regularity, thus giving a better understanding of the properties of distance-regular graphs. More precisely, we characterize  $m$ -partially distance-regular graphs and  $j$ -punctually eigenspace distance-regular graphs by using their spectra. Our results can also be seen as a generalization of the so-called spectral excess theorem for distance-regular graphs, and they lead to a dual version of it.

Keywords: Distance-regular graph, Distance matrices, Eigenvalues, Idempotents, Local spectrum, Predistance polynomials

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## 1 Preliminaries

Almost distance-regular graphs, recently studied in the literature, are graphs which share some, but not necessarily all, of the regularity properties that characterize distance-regular graphs. Two examples of the former are partially distance-regular graphs [14] and  $m$ -walk-regular graphs [6].

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In this paper we propose and characterize two dual concepts of almost distance-regularity, and study some cases where distance-regularity is attained. As in the theory of distance-regular graphs, the two proposed concepts lead to several duality results. Our results can also be seen as a generalization of the so-called spectral excess theorem for distance-regular graphs (see [9]; for short proofs, see [15, 10]). This theorem characterizes distance-regular graphs by their spectra and the average number of vertices at extremal distance. A dual version of this theorem is also derived.

We use standard concepts and results for distance-regular graphs [1, 2], spectral graph theory [4, 12], and spectral and algebraic characterizations of distance-regular graphs [8]. Moreover, for some more details and other concepts of almost distance-regularity (such as distance-polynomial and partially distance-regular graphs), we refer the reader to our recent paper [5]. In what follows, we recall the main concepts, terminology, and results involved.

Let  $\Gamma$  be a simple, connected,  $\delta$ -regular graph, with vertex set  $V$ , order  $n = |V|$ , and adjacency matrix  $\mathbf{A}$ . The *distance* between two vertices  $u$  and  $v$  is denoted by  $\text{dist}(u, v)$ , so the *diameter* of  $\Gamma$  is  $D = \max_{u, v \in V} \text{dist}(u, v)$ . The set of vertices at distance  $i$  from a given vertex  $u \in V$  is denoted by  $\Gamma_i(u)$ , for  $i = 0, 1, \dots, D$ . The *distance- $i$  graph*  $\Gamma_i$  is the graph with vertex set  $V$  and where two vertices  $u$  and  $v$  are adjacent if and only if  $\text{dist}(u, v) = i$  in  $\Gamma$ . Its adjacency matrix  $\mathbf{A}_i$  is usually referred to as the *distance- $i$  matrix* of  $\Gamma$ . The spectrum of  $\Gamma$  is denoted by  $\text{sp } \Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$ , where the different eigenvalues of  $\Gamma$  are in decreasing order,  $\lambda_0 > \lambda_1 > \dots > \lambda_d$ , and the superscripts stand for their multiplicities  $m_i = m(\lambda_i)$ .

### 1.1 The predistance and preidempotent polynomials

From the spectrum of  $\Gamma$ , we consider the *predistance polynomials*  $\{p_i\}_{0 \leq i \leq d}$  which are orthogonal with respect to the following scalar product in  $\mathbb{R}_d[x]$ :

$$\langle f, g \rangle_\Delta = \frac{1}{n} \text{tr}(f(\mathbf{A})g(\mathbf{A})) = \frac{1}{n} \sum_{i=0}^d m_i f(\lambda_i) g(\lambda_i), \quad (1)$$

and which satisfy  $\deg p_i = i$  and  $\langle p_i, p_j \rangle_\Delta = \delta_{ij} p_i(\lambda_0)$ , for all  $i, j = 0, 1, \dots, d$ . For more details, see [9]. Like every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence of the form

$$xp_i = \beta_{i-1} p_{i-1} + \alpha_i p_i + \gamma_{i+1} p_{i+1}, \quad i = 0, 1, \dots, d, \quad (2)$$

with  $\beta_{-1} = \gamma_{d+1} = 0$ . Some basic properties of these coefficients, such as  $\alpha_i + \beta_i + \gamma_i = \lambda_0$  for  $i = 0, 1, \dots, d$ , and  $\beta_i n_i = \gamma_{i+1} n_{i+1} \neq 0$  for

$i = 0, 1, \dots, d-1$ , where  $n_i = \|p_i\|_{\Delta}^2 = p_i(\lambda_0)$ , can be found in [3]. Let  $\omega_i$  be the leading coefficient of  $p_i$ . Then, from the above recurrence and since  $p(0) = 1$ , it is immediate that  $\omega_i = (\gamma_1 \gamma_2 \cdots \gamma_i)^{-1}$  for  $i = 1, \dots, d$ .

For any graph, the sum of all the predistance polynomials gives the *Hoffman polynomial*  $H$  satisfying  $H(\lambda_i) = n\delta_{0i}$ ,  $i = 0, 1, \dots, d$ , which characterizes regular graphs via the condition  $H(\mathbf{A}) = \mathbf{J}$ , the all-1 matrix [13]. Note that the leading coefficient  $\omega_d$  of  $H$  (and also of  $p_d$ ) is  $\omega_d = n/\pi_0$ .

From the predistance polynomials, we define the so-called *preidempotent polynomials*  $q_j$ ,  $j = 0, 1, \dots, d$ , by

$$q_j(\lambda_i) = \frac{m_j}{n_i} p_i(\lambda_j), \quad i = 0, 1, \dots, d,$$

which are orthogonal with respect to the scalar product

$$\langle f, g \rangle_{\mathbf{A}} = \frac{1}{n} \text{tr}(f\{\mathbf{A}\}g\{\mathbf{A}\}) = \frac{1}{n} \sum_{i=0}^d n_i f(\lambda_i) g(\lambda_i), \quad (3)$$

where  $f\{\mathbf{A}\} = \frac{1}{\sqrt{n}} \sum_{i=0}^d f(\lambda_i) p_i(\mathbf{A})$ . Note that, since  $q_j(\lambda_0) = m_j$ , the duality between the two scalar products (1) and (3) and their associated polynomials is made apparent by writing

$$\langle p_i, p_j \rangle_{\Delta} = \frac{1}{n} \sum_{l=0}^d m_l p_i(\lambda_l) p_j(\lambda_l) = \delta_{ij} n_i, \quad i, j = 0, 1, \dots, d, \quad (4)$$

$$\langle q_i, q_j \rangle_{\mathbf{A}} = \frac{1}{n} \sum_{l=0}^d n_l q_i(\lambda_l) q_j(\lambda_l) = \delta_{ij} m_i, \quad i, j = 0, 1, \dots, d. \quad (5)$$

## 1.2 Vector spaces, algebras and bases

Let  $\Gamma$  be a graph with diameter  $D$ , adjacency matrix  $\mathbf{A}$  and  $d+1$  distinct eigenvalues. We consider the vector spaces  $\mathcal{A} = \mathbb{R}_d[\mathbf{A}] = \text{span}\{\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^d\}$  and  $\mathcal{D} = \text{span}\{\mathbf{I}, \mathbf{A}, \mathbf{A}_2, \dots, \mathbf{A}_D\}$ , with dimensions  $d+1$  and  $D+1$ , respectively. Then,  $\mathcal{A}$  is an algebra with the ordinary product of matrices, known as the *adjacency algebra*, with orthogonal bases  $A_p = \{p_0(\mathbf{A}), p_1(\mathbf{A}), p_2(\mathbf{A}), \dots, p_d(\mathbf{A})\}$  and  $A_{\lambda} = \{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_d\}$ , where the matrices  $\mathbf{E}_i$ ,  $i = 0, 1, \dots, d$ , corresponding to the orthogonal projections onto the eigenspaces, are the (*principal*) *idempotents* of  $\mathbf{A}$ . Besides, since  $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^D$  are linearly independent, we have that  $\dim \mathcal{A} = d+1 \geq D+1$  and, therefore, we always have  $D \leq d$  [1]. Moreover,  $\mathcal{D}$  forms an algebra with the entrywise or Hadamard product of matrices, defined by

$(\mathbf{X} \circ \mathbf{Y})_{uv} = \mathbf{X}_{uv} \mathbf{Y}_{uv}$ . We call  $\mathcal{D}$  the *distance  $\circ$ -algebra*, which has orthogonal basis  $D_\lambda = \{\mathbf{I}, \mathbf{A}, \mathbf{A}_2, \dots, \mathbf{A}_d\}$ .

From now on, we work with the vector space  $\mathcal{T} = \mathcal{A} + \mathcal{D}$ , and relate the distance- $i$  matrices  $\mathbf{A}_i \in \mathcal{D}$  to the matrices  $p_i(\mathbf{A}) \in \mathcal{A}$ . Note that  $\mathbf{I}$ ,  $\mathbf{A}$ , and  $\mathbf{J}$  are matrices in  $\mathcal{A} \cap \mathcal{D}$  since  $\mathbf{J} = H(\mathbf{A}) \in \mathcal{A}$ . Recall that  $\mathcal{A} = \mathcal{D}$  if and only if  $\Gamma$  is distance-regular (see [1, 2]). In this case, we have  $D = d$ , and the predistance polynomials become the *distance polynomials* satisfying  $\mathbf{A}_i = p_i(\mathbf{A})$ . In  $\mathcal{T}$ , we consider the following scalar product:

$$\langle \mathbf{R}, \mathbf{S} \rangle = \frac{1}{n} \text{tr}(\mathbf{R}\mathbf{S}) = \frac{1}{n} \text{sum}(\mathbf{R} \circ \mathbf{S}), \quad (6)$$

where  $\text{sum}(\mathbf{M})$  denotes the sum of all entries of  $\mathbf{M}$ . Observe that the factor  $1/n$  assures that  $\|\mathbf{I}\|^2 = 1$ , whereas  $\|\mathbf{J}\|^2 = n$ . Note also that the *average degree* of  $\Gamma_i$  is  $\bar{\delta}_i = \|\mathbf{A}_i\|^2$  and the *average multiplicity* of  $\lambda_j$  is  $\bar{m}_j = \frac{m_j}{n} = \|\mathbf{E}_j\|^2$ . According to (1), this scalar product of matrices satisfies  $\langle f(\mathbf{A}), g(\mathbf{A}) \rangle = \langle f, g \rangle_\Delta$ .

## 2 Two dual approaches to almost distance-regularity

Here we limit ourselves to the case of graphs with spectrally maximum diameter (or the ‘non-degenerate’ case)  $D = d$ . Consequently, we will use indiscriminately the two symbols,  $D$  and  $d$ , depending on what we are referring to. In this context, let us consider the following two definitions of almost distance-regularity:

**Definition 2.1** *For a given  $i$ ,  $0 \leq i \leq D$ , a graph  $\Gamma$  is  $i$ -punctually distance-regular when there exist constants  $p_{ji}$  such that*

$$\mathbf{A}_i \mathbf{E}_j = p_{ji} \mathbf{E}_j \quad (7)$$

*for every  $j = 0, 1, \dots, d$ ; and  $\Gamma$  is  $m$ -partially distance-regular when it is  $i$ -punctually distance-regular for all  $i \leq m$ .*

**Definition 2.2** *For a given  $j$ ,  $0 \leq j \leq d$ , a graph  $\Gamma$  is  $j$ -punctually eigenspace distance-regular when there exist constants  $q_{ij}$  such that*

$$\mathbf{E}_j \circ \mathbf{A}_i = q_{ij} \mathbf{A}_i \quad (8)$$

*for every  $i = 0, 1, \dots, D$ ; and  $\Gamma$  is  $m$ -partially eigenspace distance-regular when it is  $j$ -punctually eigenspace distance-regular for all  $j \leq m$ .*

Notice that the concepts of  $D$ -partial distance-regularity and  $d$ -partial eigenspace distance-regularity coincide with the known dual definitions of distance-regularity (see [2]).

Some basic characterizations of punctual distance-regularity, in terms of the distance matrices and the idempotents, were given in [5].

**Proposition 2.3 ([5])** *Let  $D = d$ . Then,  $\Gamma$  is  $i$ -punctually distance-regular if and only if any of the following conditions holds:*

- (a1)  $\mathbf{A}_i \in \mathcal{A}$ ,
- (a2)  $p_i(\mathbf{A}) \in \mathcal{D}$ ,
- (a3)  $\mathbf{A}_i = p_i(\mathbf{A})$ .

Following the duality between Definitions 2.1 and 2.2, it seems natural to conjecture the dual of this proposition: A graph  $\Gamma$  is  $j$ -punctually eigenspace distance-regular if and only if any of the following conditions is satisfied:

- (b1)  $\mathbf{E}_j \in \mathcal{D}$ ,
- (b2)  $q_j[\mathbf{A}] \in \mathcal{A}$ ,
- (b3)  $\mathbf{E}_j = q_j[\mathbf{A}]$ ,

where  $f[\mathbf{A}] = \frac{1}{n} \sum_{i=0}^d f(\lambda_i) \mathbf{A}_i$ . However, although (b1) is clearly equivalent to Definition 2.2 and (b3)  $\Rightarrow$  (b1), (b2), until now we have not been able to prove any of the other equivalences and we leave them as conjectures.

In order to derive some new characterizations of punctual distance-regularity, besides the already defined  $\bar{\delta}_i$  and  $\bar{m}_j$ , we consider the following average numbers:

- The *average crossed local multiplicities* are

$$\bar{m}_{ij} = \frac{1}{n\bar{\delta}_i} \sum_{\text{dist}(u,v)=i} m_{uv}(\lambda_j) = \frac{\langle \mathbf{E}_j, \mathbf{A}_i \rangle}{\|\mathbf{A}_i\|^2}, \quad (9)$$

where  $m_{uv}(\lambda_j) = (\mathbf{E}_j)_{uv}$  are the *crossed local multiplicities*.

- The *average number of shortest  $i$ -paths from a vertex* is

$$\bar{P}_i = \frac{1}{n} \sum_{u \in V} P_i(u) = \frac{1}{n} \text{sum}(\mathbf{A}^i \circ \mathbf{A}_i) = \langle \mathbf{A}^i, \mathbf{A}_i \rangle = \frac{1}{\omega_i} \langle p_i(\mathbf{A}), \mathbf{A}_i \rangle, \quad (10)$$

where  $P_i(u)$  denotes the number of shortest paths from a vertex  $u$  to the vertices in  $\Gamma_i(u)$  and  $\omega_i = (\gamma_1\gamma_2\cdots\gamma_i)^{-1}$  is the leading coefficient of  $p_i$ ,  $i = 1, \dots, d$ .

- The average number of shortest  $i$ -paths is

$$\bar{a}_i^{(i)} = \frac{1}{n\bar{\delta}_i} \text{sum}(\mathbf{A}^i \circ \mathbf{A}_i) = \frac{\bar{P}_i}{\bar{\delta}_i}. \quad (11)$$

**Proposition 2.4** *Let  $\Gamma$  be a graph with predistance polynomials  $p_i$  and recurrence coefficients  $\gamma_i, \alpha_i, \beta_i$ ,  $i = 0, 1, \dots, d$ . Then,  $\Gamma$  is  $i$ -punctually distance-regular if and only if any of the following equalities holds:*

$$(a1) \quad \frac{1}{\bar{\delta}_i} = \sum_{j=0}^d \frac{\bar{m}_{ij}^2}{\bar{m}_j}.$$

$$(a2) \quad \bar{P}_i = \frac{1}{\omega_i} \sqrt{p_i(\lambda_0)\bar{\delta}_i} = \sqrt{\beta_0\beta_1\cdots\beta_{i-1}\bar{\delta}_i\gamma_i\gamma_{i-1}\cdots\gamma_1}.$$

$$(a3) \quad \omega_i \bar{a}_i^{(i)} = 1 \quad \text{and} \quad \bar{\delta}_i = p_i(\lambda_0).$$

Moreover,  $\Gamma$  is  $j$ -punctually eigenspace distance-regular if and only if

$$(b1) \quad \bar{m}_j = \sum_{i=0}^D \bar{\delta}_i \bar{m}_{ij}^2.$$

**Proof.** (a1) This is a result from [5].

(a2) From (10) and the Cauchy-Schwarz inequality, we get

$$\omega_i \bar{P}_i = \langle p_i(\mathbf{A}), \mathbf{A}_i \rangle \leq \|p_i(\mathbf{A})\| \|\mathbf{A}_i\| = \sqrt{p_i(\lambda_0)\bar{\delta}_i} = \sqrt{\frac{\beta_0\beta_1\cdots\beta_{i-1}}{\gamma_1\gamma_2\cdots\gamma_i}\bar{\delta}_i}. \quad (12)$$

Moreover, equality occurs if and only if the matrices  $p_i(\mathbf{A})$  and  $\mathbf{A}_i$  are proportional, which is equivalent to  $\Gamma$  being  $i$ -punctually distance-regular by Proposition 2.3.

(a3) From (11) and (12) we have that  $\omega_i \bar{a}_i^{(i)} \leq \sqrt{p_i(\lambda_0)/\bar{\delta}_i}$ , with equality if and only if  $\Gamma$  is  $i$ -punctually distance-regular. Thus, if the conditions in (a3) hold,  $\Gamma$  satisfies the claimed property. Conversely, if  $\Gamma$  is  $i$ -punctually distance-regular, both equalities in (a3) are simple consequences of  $p_i(\mathbf{A}) = \mathbf{A}_i$ . Indeed, the first one comes from considering the  $uv$ -entries, with  $\text{dist}(u, v) = i$ , in the above matrix equation, whereas the second one is obtained by taking square norms.

(b1) From (9), we find that the orthogonal projection of  $\mathbf{E}_j$  on  $\mathcal{D}$  is  $\widehat{\mathbf{E}}_j = \sum_{i=0}^D \overline{m}_{ij} \mathbf{A}_i$ . Now, from  $\|\widehat{\mathbf{E}}_j\|^2 \leq \|\mathbf{E}_j\|^2$  we get

$$\sum_{i=0}^D \overline{m}_{ij}^2 \|\mathbf{A}_i\|^2 = \sum_{i=0}^D \overline{\delta}_i \overline{m}_{ij}^2 \leq \overline{m}_j$$

and, in the case of equality, Definition 2.2 applies with  $q_{ij} = \overline{m}_{ij}$ .  $\square$

Notice the duality between (a1) and (b1) with  $\frac{1}{\overline{\delta}_i}$  and  $\overline{m}_j$ .

Now, let us consider the more global concept of partial distance-regularity. In this case, we also have the following new result where, for a given  $0 \leq i \leq d$ ,  $s_i = \sum_{j=0}^i p_j$ ,  $t_i = H - s_{i-1} = \sum_{j=i}^d p_j$ ,  $\mathbf{S}_i = \sum_{j=0}^i \mathbf{A}_j$ , and  $\mathbf{T}_i = \mathbf{J} - \mathbf{S}_{i-1} = \sum_{j=i}^d \mathbf{A}_j$ .

**Proposition 2.5** *A graph  $\Gamma$  is  $m$ -partially distance-regular if and only if any of the following conditions holds:*

- (a1)  $\Gamma$  is  $i$ -punctually distance-regular for  $i = m, m-1, \dots, \max\{2, 2m-d\}$ .
- (a2)  $\Gamma$  is  $m$ -punctually distance-regular and  $t_{m+1}(\mathbf{A}) \circ \mathbf{S}_m = \mathbf{O}$ .
- (a3)  $s_i(\mathbf{A}) = \mathbf{S}_i$  for  $i = m, m-1$ .

**Proof.** In all cases, the necessity is clear since  $p_i(\mathbf{A}) = \mathbf{A}_i$  for every  $0 \leq i \leq m$  (for (a2), note that  $t_{m+1}(\mathbf{A}) = \mathbf{J} - \mathbf{S}_m(\mathbf{A})$ ). Then, let us prove sufficiency. The result in (a1) is basically Proposition 3.7 in [5]. In order to prove (a2), we show by (backward) induction that  $p_i(\mathbf{A}) = \mathbf{A}_i$  and  $t_{i+1}(\mathbf{A}) \circ \mathbf{S}_i = \mathbf{O}$  for  $i = m, m-1, \dots, 0$ . By assumption, these equations are valid for  $i = m$ . Suppose now that  $p_i(\mathbf{A}) = \mathbf{A}_i$  and  $t_{i+1}(\mathbf{A}) \circ \mathbf{S}_i = \mathbf{O}$  for some  $i > 0$ . Then,  $t_i(\mathbf{A}) \circ \mathbf{S}_i = \mathbf{A}_i$  and, multiplying both terms by  $\mathbf{S}_{i-1}$  (with the Hadamard product), we get  $t_i(\mathbf{A}) \circ \mathbf{S}_{i-1} = \mathbf{O}$ . So, what remains is to show that  $p_{i-1}(\mathbf{A}) = \mathbf{A}_{i-1}$ . To this end, let us consider the following three cases:

- (i) For  $\text{dist}(u, v) > i-1$ , we have  $(p_{i-1}(\mathbf{A}))_{uv} = 0$ .
- (ii) For  $\text{dist}(u, v) = i-1$ , we have  $(t_{i+1}(\mathbf{A}))_{uv} = 0$ , so  $(p_{i-1}(\mathbf{A}))_{uv} = (s_{i-1}(\mathbf{A}))_{uv} = (s_{i-1}(\mathbf{A}))_{uv} + (\mathbf{A}_i)_{uv} = (s_i(\mathbf{A}))_{uv} = 1 - (t_{i+1}(\mathbf{A}))_{uv} = 1$ .
- (iii) For  $\text{dist}(u, v) < i-1$ , we use the recurrence (2) to write

$$xp_i = \sum_{j=i}^d xp_j = \sum_{j=i}^d (\beta_{j-1}p_{j-1} + \alpha_j p_j + \gamma_{j+1}p_{j+1})$$

$$\begin{aligned}
&= \beta_{i-1}p_{i-1} - \gamma_i p_i + \sum_{j=i}^d (\alpha_j + \beta_j + \gamma_j)p_j \\
&= \beta_{i-1}p_{i-1} - \gamma_i p_i + \delta t_i,
\end{aligned}$$

which gives

$$\mathbf{A}t_i(\mathbf{A}) = \beta_{i-1}p_{i-1}(\mathbf{A}) - \gamma_i \mathbf{A}_i + \delta t_i(\mathbf{A}).$$

Then, since  $(t_i(\mathbf{A}))_{uv} = (\mathbf{A}_i)_{uv} = 0$  and  $\beta_{i-1} \neq 0$ , we get

$$(p_{i-1}(\mathbf{A}))_{uv} = \frac{1}{\beta_{i-1}}(\mathbf{A}t_i(\mathbf{A}))_{uv} = \frac{1}{\beta_{i-1}} \sum_{w \in \Gamma(u)} (t_i(\mathbf{A}))_{wv} = 0,$$

because  $\text{dist}(v, w) \leq \text{dist}(v, u) + \text{dist}(u, w) \leq i - 1$  for the relevant  $w$ .

From (i), (ii), and (iii), we have that  $p_{i-1}(\mathbf{A}) = \mathbf{A}_{i-1}$ , so by induction  $\Gamma$  is  $m$ -partially distance-regular, and the sufficiency of (a2) is proven. Finally, the sufficiency of (a3) follows from that of (a2) because  $s_i(\mathbf{A}) = \mathbf{S}_i$  for every  $i \in \{m-1, m\}$  implies that  $p_m(\mathbf{A}) = (s_m - s_{m-1})(\mathbf{A}) = \mathbf{S}_m - \mathbf{S}_{m-1} = \mathbf{A}_m$  and  $t_{m+1}(\mathbf{A}) \circ \mathbf{S}_m = (\mathbf{J} - s_m(\mathbf{A})) \circ \mathbf{S}_m = (\mathbf{J} - \mathbf{S}_m) \circ \mathbf{S}_m = \mathbf{O}$ .  $\square$

Given some vertex  $u$  and an integer  $i \leq \text{ecc}(u)$ , we denote by  $N_i(u)$  the  $i$ -neighborhood of  $u$ , which is the set of vertices that are at distance at most  $i$  from  $u$ . In [8] it was proved that  $s_i(\lambda_0)$  is upper bounded by the harmonic mean of the numbers  $|N_i(u)|$  and equality is attained if and only if  $s_i(\mathbf{A}) = \mathbf{S}_i$ . A direct consequence of this property and Proposition 2.5(a3) is the following characterization.

**Theorem 2.6** *A graph  $\Gamma$  is  $m$ -partially distance-regular if and only if, for every  $i \in \{m-1, m\}$ ,*

$$s_i(\lambda_0) = \frac{n}{\sum_{u \in V} |N_i(u)|^{-1}}.$$

### 3 Distance-regular graphs

Let us particularize our results to the case of distance-regular graphs. With this aim, we use the following theorem giving some known characterizations.

**Theorem 3.1** ([7, 11]) *A graph  $\Gamma$  with  $d+1$  distinct eigenvalues and diameter  $D = d$  is distance-regular if and only if any of the following statements is satisfied:*



- (a)  $\Gamma$  is  $D$ -punctually distance-regular.
- (b)  $\Gamma$  is  $j$ -punctually eigenspace distance-regular for  $j = 1, d$ .

In fact, notice that (a) corresponds to any of the conditions in Proposition 2.5 with  $m = d$ . Moreover, the duality between (a) and (b) is made apparent when they are stated as follows:

- (a)  $\mathbf{A}_0(= \mathbf{I}), \mathbf{A}_1(= \mathbf{A}), \mathbf{A}_D \in \mathcal{A}$ ;
- (b)  $\mathbf{E}_0(= \frac{1}{n}\mathbf{J}), \mathbf{E}_1, \mathbf{E}_d \in \mathcal{D}$ .

Then, by using Theorem 3.1 and Proposition 2.4(a1) and (b1), and Theorem 2.6 (with  $m = d$ ), we have the spectral excess theorem [9] in the next condition (a), its dual form in (b), and its harmonic mean version [8, 15] in (c).

**Theorem 3.2** *A regular graph  $\Gamma$  with  $D = d$  is distance-regular if and only if any of the following equalities holds:*

- (a)  $\frac{1}{\bar{\delta}_d} = \sum_{j=0}^d \frac{\bar{m}_{dj}^2}{\bar{m}_j}$ .
- (b)  $\bar{m}_j = \sum_{i=0}^D \bar{\delta}_i \bar{m}_{ij}^2$  for  $j = 1, d$ .
- (c)  $s_{d-1}(\lambda_0) = \frac{n}{\sum_{u \in V} |N_{d-1}(u)|^{-1}}$ .

In fact, condition (a) is usually written in its equivalent form  $\bar{\delta}_d = p_d(\lambda_0)$  as, when  $i = d$ , the first condition in Proposition 2.4(a.3) always holds since

$$\bar{a}_d^{(d)} = \frac{1}{\bar{\delta}_d} \langle \mathbf{A}^d, \mathbf{A}_d \rangle = \frac{1}{\bar{\delta}_d \omega_d} \langle H(\mathbf{A}), \mathbf{A}_d \rangle = \frac{1}{\bar{\delta}_d \omega_d} \langle \mathbf{J}, \mathbf{A}_d \rangle = \frac{1}{\bar{\delta}_d \omega_d} \|\mathbf{A}_d\|^2 = \frac{1}{\omega_d}.$$

Notice also that, in (c), we do not need to impose the condition of Theorem 2.6 for  $i = d$  since  $s_d(\lambda_0) = H(\lambda_0) = N_d(u) = n$  for every  $u \in V$ .

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